12.1: Triangle Proportionality Theorem

Explain 1: Proving the Triangle Proportionality Theorem

As you saw in the Explore, when a line parallel to one side of a triangle intersects the other two sides of the triangle, the lengths of the segments are proportional.

**Triangle Proportionality Theorem**

<table>
<thead>
<tr>
<th>Theorem</th>
<th>Hypothesis</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>If a line parallel to a side of a triangle intersects the other two sides, then it divides those sides proportionally.</td>
<td>$\frac{AE}{EB} = \frac{AF}{FC}$</td>
<td></td>
</tr>
</tbody>
</table>

\[ EF \parallel BC \]
Example 1  Prove the Triangle Proportionality Theorem

Given: \( \overrightarrow{EF} \parallel \overrightarrow{BC} \)

Prove: \( \frac{AE}{EB} = \frac{AF}{FC} \)

Step 1  Show that \( \triangle AEF \sim \triangle ABC \).

Because \( \overrightarrow{EF} \parallel \overrightarrow{BC} \), you can conclude that \( \angle 1 \cong \angle 2 \) and \( \angle 3 \cong \angle 4 \) by the **Corresponding Angles** Theorem.

So, \( \triangle AEF \sim \triangle ABC \) by the **AA Similarity Criterion**.
Step 2 Use the fact that corresponding sides of similar triangles are proportional to prove that \( \frac{AE}{EB} = \frac{AF}{FC} \).

<table>
<thead>
<tr>
<th>( \frac{AB}{AE} )</th>
<th>( \frac{AC}{AF} )</th>
<th>Corresponding sides are proportional.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{AE + EB}{AE} )</td>
<td>( \frac{AF + FC}{AF} )</td>
<td>Segment Addition Postulate</td>
</tr>
<tr>
<td>( 1 + \frac{EB}{AB} )</td>
<td>( 1 + \frac{FC}{AF} )</td>
<td>Use the property that ( \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c} ).</td>
</tr>
<tr>
<td>( \frac{EB}{AE} )</td>
<td>( \frac{FC}{AF} )</td>
<td>Subtract 1 from both sides.</td>
</tr>
<tr>
<td>( \frac{AE}{EB} )</td>
<td>( \frac{AF}{FC} )</td>
<td>Take the reciprocal of both sides.</td>
</tr>
</tbody>
</table>

**Reflect**

4. Explain how you conclude that \( \triangle AEF \sim \triangle ABC \) without using \( \angle 3 \) and \( \angle 4 \).
Example 2

Find the length of each segment.

It is given that $\overline{XY} \parallel \overline{BC}$ so $\frac{AX}{XB} = \frac{AY}{YC}$ by the Triangle Proportionality Theorem.

Substitute 9 for $AX$, 4 for $XB$, and 10 for $AY$.

Then solve for $CY$.

$$\frac{9}{4} = \frac{10}{CY}$$

Take the reciprocal of both sides.

$$\frac{4}{9} = \frac{CY}{10}$$

Next, multiply both sides by 10.

$$10(\frac{4}{9}) = (\frac{CY}{10})10 \quad \rightarrow \quad \frac{40}{9} = CY, \text{ or } 4\frac{4}{9} = CY$$
Find $PN$. 

It is given that $PQ \parallel LM$, so \( \frac{NQ}{QM} = \frac{NP}{PL} \) by the Triangle Proportionality Theorem. 

Substitute 5 for $NQ$, 2 for $QM$, and 3 for $PL$. 

\[ \frac{5}{2} = \frac{NP}{3} \]

Multiply both sides by 3: 

\[ 3 \left( \frac{5}{2} \right) = 3 \left( \frac{NP}{3} \right) \rightarrow \]

\[ NP = \frac{15}{2} \text{ or } 7\frac{1}{2} \]
Find the length of each segment.

5. \( \overline{DG} \)

\[
\frac{EC}{CF} = \frac{ED}{DG};
\]

\[
\frac{32}{24} = \frac{40}{DG};
\]

\[
\frac{24}{32} = \frac{DG}{40};
\]

\[
40 \left( \frac{24}{32} \right) = DG;
\]

\[
DG = \frac{960}{32} = 30
\]

6. \( \overline{RN} \)

\[
\frac{MR}{RN} = \frac{MQ}{QP};
\]

\[
\frac{10}{RN} = \frac{8}{5};
\]

\[
\frac{RN}{10} = \frac{5}{8};
\]

\[
RN = 10 \times \frac{5}{8} = \frac{25}{4} = 6 \frac{1}{4}
\]
Proving the Converse of the Triangle Proportionality Theorem

The converse of the Triangle Proportionality Theorem is also true.

<table>
<thead>
<tr>
<th>Converse of the Triangle Proportionality Theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Theorem</strong></td>
</tr>
<tr>
<td>If a line divides two sides of a triangle proportionally, then it is parallel to the third side.</td>
</tr>
</tbody>
</table>

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<th><strong>Hypothesis</strong></th>
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<tr>
<td>$\frac{AE}{EB} = \frac{AF}{FC}$</td>
<td>$\overrightarrow{EF} \parallel \overrightarrow{BC}$</td>
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</table>
Example 3  Prove the Converse of the Triangle Proportionality Theorem

Given: \( \frac{AE}{EB} = \frac{AF}{FC} \)

Prove: \( \overrightarrow{EF} \parallel \overrightarrow{BC} \)

**Step 1** Show that \( \triangle AEF \sim \triangle ABC \).

It is given that \( \frac{AE}{EB} = \frac{AF}{FC} \), and taking the reciprocal of both sides shows that \( \frac{EB}{AE} = \frac{FC}{AF} \). Now add 1 to both sides by adding \( \frac{AE}{AE} \) to the left side and \( \frac{AF}{AF} \) to the right side.

This gives \( \frac{AE}{AE} + \frac{EB}{AE} = \frac{AF}{AF} + \frac{FC}{AF} \).

Adding and using the Segment Addition Postulate gives \( \frac{AB}{AE} = \frac{AC}{AF} \).

Since \( \angle A \cong \angle A \), \( \triangle AEF \sim \triangle ABC \) by the **SAS Similarity** Theorem.

**Step 2** Use corresponding angles of similar triangles to show that \( \overrightarrow{EF} \parallel \overrightarrow{BC} \).

\( \angle AEF \cong \angle ABC \) and are corresponding angles.

So, \( \overrightarrow{EF} \parallel \overrightarrow{BC} \) by the **Converse of the Corresponding Angles** Theorem.
7. **Critique Reasoning** A student states that $\overline{UV}$ must be parallel to $\overline{ST}$. Do you agree? Why or why not?

Yes; because $RU = US$ and $RV = VS$, $\frac{RU}{US} = \frac{RV}{VT} = 1$.

So $\overline{UV} \parallel \overline{ST}$ by the Converse of the Triangle Proportionality Theorem.
**Example 4** Verify that the line segments are parallel.

(A) \(MN\) and \(KL\)

\[\frac{JM}{MK} = \frac{42}{21} = 2 \quad \frac{JN}{NL} = \frac{30}{15} = 2\]

Since \(\frac{JM}{MK} = \frac{JN}{NL}\), \(MN \parallel KL\) by the Converse of the Triangle Proportionality Theorem.

(B) \(DE\) and \(AB\) (Given that \(AC = 36\) cm, and \(BC = 27\) cm)

\[AD = AC - DC = 36 - 20 = 16\]

\[BE = BC - EC = 27 - 15 = 12\]

\[\frac{CD}{DA} = \frac{20}{16} = \frac{5}{4} \quad \frac{CE}{EB} = \frac{15}{12} = \frac{5}{4}\]

Since \(\frac{CD}{DA} = \frac{CE}{EB}\), \(DE \parallel AB\) by the Converse of the Triangle Proportionality Theorem.
\[ \frac{AB}{DE} = \frac{AC}{DC} = \frac{9}{5}; \text{ Possible answer: because } DE \parallel AB, \text{ corresponding angles } A \text{ and } CDE \text{ are congruent, as are corresponding angles } B \text{ and } CED. \text{ So, } \triangle ABC \sim \triangle DEC \text{ and } \frac{AB}{DE} = \frac{AC}{DC} = \frac{BC}{EC}. \]

8. **Communicate Mathematical Ideas** In \( \triangle ABC \), in the example, what is the value of \( \frac{AB}{DE} \)? Explain how you know.

\[ \overline{DE} \text{ and } \overline{AB} \quad (\text{Given that } AC = 36 \text{ cm, and } BC = 27 \text{ cm}) \]

Your Turn

9. **Verify that** \( TU \) **and** \( RS \) **are parallel.**

\[ \frac{VT}{TR} = \frac{90}{72} = \frac{5}{4}, \quad \frac{VU}{US} = \frac{67.5}{54} = \frac{135}{108} = \frac{5}{4} \]

\[ \frac{VT}{TR} = \frac{VU}{US}, \text{ so } RS \parallel TU. \]
12.3 Using Proportional Relationships

Essential Question: How can you use similar triangles to solve problems?

Explore Exploring Indirect Measurement

In this Explore, you will consider how to find heights, lengths, or distances that are too great to be measured directly, that is, with measuring tools like rulers. Indirect measurement involves using the properties of similar triangles to measure such heights or distances.
**Finding an Unknown Height**

Because $\overline{ZX} \parallel \overline{CA}$, $\angle Z \cong \angle C$. All right angles are congruent, so $\angle Y \cong \angle B$. So $\triangle XYZ \sim \triangle ABC$.

Set up proportion. 

$$\frac{AB}{XY} = \frac{BC}{YZ}$$

Substitute. 

$$\frac{h}{7.2} = \frac{1}{1.6}$$

Multiply each side by 7.2. 

$$h = 7.2 \left( \frac{1}{1.6} \right)$$

Simplify. 

$$h = 4.5$$

The tree is 4.5 meters high.
Sid is 72 inches tall. To measure a flagpole, Sid stands near the flag. Sid’s friend Miranda measures the lengths of Sid’s shadow and the flagpole’s shadow. Find the height \( h \) of the flagpole.

The triangles are similar by the AA Triangle Similarity Theorem.

Set up proportion.

\[
\frac{\text{flagpole’s height}}{\text{person’s height}} = \frac{\text{flagpole’s shadow}}{\text{person’s shadow}}
\]

Substitute.

\[
\frac{h}{72} = \frac{128}{48}
\]

Multiply each side by 72.

\[
h = 72 \left( \frac{128}{48} \right)
\]

Simplify.

\[
x = 192
\]

The flagpole is 192 inches tall.
3. Liam is 6 feet tall. To find the height of a tree, he measures his shadow and the tree’s shadow. The measurements of the two shadows are shown. Find the height $h$ of the tree.

The triangles are similar by the AA Similarity Criterion.

\[
\frac{\text{tree's height}}{\text{Liam's height}} = \frac{\text{tree's shadow}}{\text{Liam's shadow}}
\]

\[
\frac{h}{6} = \frac{28}{8} \quad h = 6 \left( \frac{28}{8} \right)
\]

\[
h = \frac{168}{8} \quad h = 21
\]

The tree is 21 feet tall.
Finding an Unknown Distance

In real-world situations, you may not be able to measure an object directly because there is a physical barrier separating you from the object. You can use similar triangles in these situations as well.

Example 2 Explain how to use the information in the figure to find the indicated distance.

A hiker wants to find the distance $d$ across a canyon. She locates points as described.

1. She identifies a landmark at $X$. She places a marker $(Y)$ directly across the canyon from $X$.

2. At $Y$, she turns 90° away from $X$ and walks 400 feet in a straight line. She places a marker $(Z)$ at this location.

3. She continues walking another 600 feet, and places a marker $(W)$ at this location.

4. She turns 90° away from the canyon and walks until the marker $Z$ aligns with $X$. She places a marker $(V)$ at this location and measures $WV$.

$\angle VWZ \cong \angle XYZ$ (All right angles are congruent) and $\angle VZW \cong \angle XZY$ (Vertical angles are congruent). So, $\triangle VWZ \sim \triangle XYZ$ by the AA Triangle Similarity Theorem.

\[
\frac{XY}{VW} = \frac{YZ}{WZ}. \text{ So } \frac{d}{327} = \frac{400}{600}, \text{ or } \frac{d}{327} = \frac{2}{3}
\]

Then $d = 327 \left( \frac{2}{3} \right) = 218$. The distance across the canyon is 218 feet.
To find the distance \( d \) across the gorge, a student identifies points as shown in the figure. Find \( d \).

\[ \triangle JKL \sim \triangle NML \] by the AA Triangle Similarity Theorem.

\[ \frac{JK}{NM} = \frac{KL}{ML} = \frac{d}{35} \]

\[ \frac{24}{42} \]

\[ d = 35 \cdot \frac{24}{42} = 35 \cdot \frac{4}{7} \]

\[ d = \frac{140}{7} = 20 \]

The distance across the gorge is **20 meters**.

**Reflect**

4. In the example, why is \( \angle JKL \cong \angle NLM \)?

\( \angle JKL \cong \angle NLM \) are vertical angles and vertical angles are congruent.
5. To find the distance $d$ across a stream, Levi located points as shown in the figure. Use the given information to find $d$.

$\triangle ABC \sim \triangle DEC$ by the AA Triangle Similarity Theorem.

\[
\frac{AB}{DE} = \frac{BC}{EC}; \quad \frac{d}{12} = \frac{12}{6}; \quad d = 12 \left( \frac{12}{6} \right) = 24
\]

$d = 24$ meters